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Symmetries, first integrals and the inverse problem of Lagrangian mechanics

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Abstract. The paper deals with the following question: given a symmetry vector field Y of a system of second-order ordinary differential equations, and an associated constant of the motion F, is it possible to find a Lagrangian L for the system, such that Y becomes a Noether symmetry with respect to L, and F its implied Noether constant? It is shown that for one degree of freedom systems the answer to this question is affirmative. In addition, attention is paid to the construction of a suitable constant of the motion F for given symmetry Y and vice versa. Several examples are discussed.

1. Introduction

In recent papers we have presented a comparative review of different approaches to Noether's theorem in classical mechanics (Sarlet and Cantrijn 1981a), as well as some generalisations (Sarlet and Cantrijn 1981b). As is well known, Noether's theorem (Noether 1918) in one way or another establishes a link between invariance transformations of the action integral and constants of the motion. Most familiar is the energy integral of conservative systems, which can be related to time-translation invariance of an action principle. Often, however, there have been misconceptions about these matters, which mainly originate from the following observations. First, there are ambiguities in the possible Lagrangian descriptions of given second-order equations, and secondly, an invariance transformation of the equations of motion is not necessarily a symmetry of a given Lagrangian representation. Interesting papers in this respect are those of Havas (1973) and Marmo and Saletan (1977). This brings us to the second part of our title: the inverse problem of Lagrangian mechanics, i.e. the question of how to construct (if possible) a Lagrangian for given second-order (ordinary) differential equations. For a list of references, and a historical coverage of this topic, see Santilli (1978).

In reviewing different versions of Noether's theorem, we have expressed a preference for those versions in which there is some kind of uniqueness in the relationship between symmetry generators of Noether type Y and constants of the motion F (in fact a precise one-to-one correspondence between equivalence classes of Y's and F's). Of course, such a statement of uniqueness is strictly related to a given, fixed Lagrangian description. It is even a necessary prerequisite if one wants to study subsequently the influence of passing from one Lagrangian description to another.

If we schematically represent Noether's theorem and its converse by the full arrows in figure 1, we immediately face a complementary problem (represented by the broken lines), namely whether the knowledge of a symmetry generator Y and corresponding constant of the motion F can help us to construct a Lagrangian for a given system.

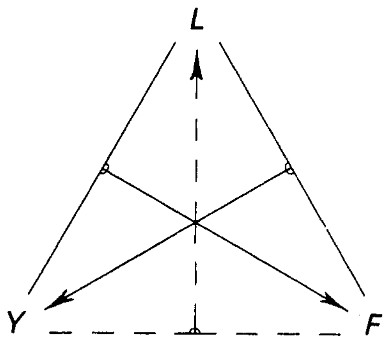

Figure 1. Full lines represent Noether's theorem with converse; broken lines suggest the problem discussed here: for given symmetry and related first integral, find a suitable Lagrangian.

This question is the subject of the present paper, and will be described in more precise terms in the next section. In studying this question, one quickly observes that the case of one degree of freedom substantially differs from the multiple degree of freedom case, and therefore deserves a separate treatment. The fact that the inverse problem for systems with one degree of freedom always has solutions is already a sufficient motivation for this. Moreover, there is ample material (on systems with one degree of freedom) with which our results can be compared. In some sense our present results, for example, will generalise those of Kobussen (1979) and Sarlet (1978). It also suits us here to recall recent studies related to the complete (point) symmetry group of the harmonic oscillator and other linear equations (Wulfman and Wybourne 1976, Lutzky 1978, Leach 1980a, b, c). These studies reveal an eight-parameter group of point symmetries. A five-parameter subgroup consists of Noether symmetries (with respect to the usual Lagrangian), while the three complementary symmetries are then referred to as non-Noether symmetries. The main theorem of the present paper will show that these 'non-Noether' symmetries are actually Noether symmetries, but with respect to an 'unusual' Lagrangian description.

The plan of the paper is as follows. Section 2 recalls the appropriate form of Noether's theorem on which we will rely, as well as some related results. An example, treated in § 3, serves as an introduction to the main theorem which is proved in § 4. This section further contains results for the cases that only Y or only F are known. Section 5 presents a number of other examples, which are selected to clarify links with earlier quoted papers. Problems related to the generalisation for systems with multiple degree of freedom are briefly discussed in § 6.

2. Preliminaries

The contents of this section are general in the sense that everything applies to systems with multiple degree of freedom. We make use of elementary operations on differential forms, with notations which are customary in a differential geometric context. The further analysis, however, is entirely local and not very geometrical in character. For general definitions related to the calculus of differential forms and vector fields, one can consult e.g. Abraham and Marsden (1978) or Godbillon (1969), but an intuitive sketch like the one in Sarlet and Cantrijn (1981a) will amply suffice for the scope of this paper.

Consider first a general system of n second-order ordinary differential equations, written in normal form,

$$\ddot{q}^i = \Lambda^i(t, q, \dot{q}), \qquad i = 1, \ldots, n. \tag{1}$$

Passing to an equivalent first-order system in the usual way, it can be associated with the following vector field in $(t, q, \dot{q})$ space:

$$\Gamma = \partial/\partial t + \dot{q}^i\, \partial/\partial q^i + \Lambda^i(t, q, \dot{q})\, \partial/\partial \dot{q}^i. \tag{2}$$

If system (1) is derivable from a Lagrangian $L(t, q, \dot{q})$, we can consider the one-form

$$\theta = L\, \mathrm{d}t + (\partial L/\partial \dot{q}^i)(\mathrm{d}q^i - \dot{q}^i\, \mathrm{d}t), \tag{3}$$

often called Poincaré–Cartan form. Saying that (1) is a system of Euler–Lagrange equations is then equivalent to saying that Γ is a characteristic vector field of $\mathrm{d}\theta$, i.e.

$$i_\Gamma\, \mathrm{d}\theta = 0, \tag{4}$$

where i_Γ stands for the inner product of forms with Γ.

A vector field Y generates an infinitesimal symmetry transformation of system (1), if

$$L_Y\Gamma \equiv [Y, \Gamma] = g\Gamma, \tag{5}$$

where g is some function of t, q, $\dot{q}$, and L_Y is the Lie derivative. If Y is represented in the form

$$Y = \tau(t, q, \dot{q})\, \partial/\partial t + \xi^i(t, q, \dot{q})\, \partial/\partial q^i + \eta^i(t, q, \dot{q})\, \partial/\partial \dot{q}^i, \tag{6}$$

the symmetry requirement (5) means that

$$\eta^i = \Gamma(\xi^i) - \dot{q}^i\Gamma(\tau) \equiv \Gamma(\xi^i - \dot{q}^i\tau) + \Lambda^i\tau, \tag{7}$$

$$Y(\Lambda^i) = \Gamma(\eta^i) - \Lambda^i\Gamma(\tau). \tag{8}$$

Multiples of Γ are 'trivial symmetries' of (1). Therefore we can call two symmetries Y_1 and Y_2 *equivalent* if they differ by such a multiple,

$$Y_2 = Y_1 + h\Gamma \qquad (h \text{ a function}). \tag{9}$$

Choosing $h = -\tau$ for a symmetry like (6), we can work with the equivalent symmetry (again denoted by Y),

$$Y = \mu^i(t, q, \dot{q})\, \partial/\partial q^i + \nu^i(t, q, \dot{q})\, \partial/\partial \dot{q}^i, \tag{10}$$

where

$$\mu^i = \xi^i - \dot{q}^i\tau, \tag{11}$$

and the symmetry requirements (7) and (8) reduce to

$$\nu^i = \Gamma(\mu^i), \tag{12}$$

$$Y(\Lambda^i) = \Gamma(\nu^i). \tag{13}$$

For systems (1) which are derivable from a Lagrangian L, we can consider a subclass of symmetries, called Noether symmetries, defined as follows.

Definition. Let Γ satisfy condition (4), with θ defined by (3). Then Y is a Noether symmetry with respect to the Lagrangian L if and only if

$$L_Y \, \mathrm{d}\theta = 0. \tag{14}$$

Condition (14), in this purely local context, of course implies $L_Y\theta = \mathrm{d}f$ for some function f, and one can easily prove the following version of Noether's theorem (with converse).

Theorem (Noether). Let Γ satisfy (4) for some Lagrangian L; then

(i) to each Noether symmetry Y (with respect to L) corresponds a constant of the motion F, which is uniquely determined (up to a trivial constant) by

$$i_Y \, \mathrm{d}\theta = \mathrm{d}F; \tag{15}$$

(ii) to each constant of the motion F of Γ corresponds a Noether symmetry Y, which is unique up to an equivalence of type (9);

(iii) F is in addition an invariant of Y, i.e.

$$Y(F) = 0. \tag{16}$$

The statements (i) and (ii) now give a precise meaning to the full arrows of figure 1, while the property (iii) is particularly useful for the type of problem we want to investigate here: given a general symmetry Y of Γ, and a constant of the motion F, can we find a Lagrangian L for system (1), such that Y and F become interrelated through Noether's theorem with respect to that L? Obviously by (iii), a necessary prerequisite for a positive answer to that question is that Y and F satisfy (16).

In studying different versions of Noether's theorem (Sarlet and Cantrijn 1981a), we have proved some kind of equivalence between the approach summarised above and the original approach via invariance of the action integral. The precise formulation of this equivalence will be useful in proving the main theorem of § 4, and therefore is recalled here in a slightly different form as a lemma.

Lemma. Let θ be a one-form of type (3), F a function, and Y a vector field of type (10). Then

$$\text{(I)} \qquad \left.\begin{array}{l} \langle \partial/\partial \dot{q}^j, i_Y \, \mathrm{d}\theta - \mathrm{d}F \rangle = 0 \\ \Gamma(F) = 0 \\ \nu^i = \Gamma(\mu^i) \end{array}\right\} \Leftrightarrow i_Y \, \mathrm{d}\theta = \mathrm{d}F,$$

where $\langle , \rangle$ stands for the pairing between one-forms and vector fields.

Finally, we want to re-emphasise the importance of the first of equations (I), which in a local coordinate expression reads

$$(\partial^2 L/\partial \dot{q}^i \, \partial \dot{q}^j)\mu^j = -\partial F/\partial \dot{q}^i. \tag{17}$$

When L and F are given, equation (17) enables us to compute immediately the Noether symmetry Y corresponding to F. Indeed, μ^i being determined by (17), the remaining components ν^i of Y follow directly from one of the symmetry requirements, namely (12). It is clear now that the same relation (17) can also play a crucial role in the present context, because it contains in one and the same formula not only ingredients from the symmetry Y and the constant F, but also the factor $\partial^2 L/\partial \dot{q}^i \, \partial \dot{q}^j$, which after all is the multiplier which should transform equation (1) to an Euler–Lagrange form. For more details about the contents of this section, see Sarlet and Cantrijn (1981a, b).

From now on, we will specialise to the case $n = 1$, and first illustrate the use of equation (17) on an example.

3. An example

Consider the simple harmonic oscillator equation,

$$\ddot{q} = -q. \tag{18}$$

As stated in the Introduction, it is well known that (18) has exactly eight point-symmetries, i.e. symmetry generators like (6) for which ξ and τ do not depend on $\dot{q}$, five of which are Noether symmetries with respect to the usual Lagrangian $\frac{1}{2}(\dot{q}^2 - q^2)$. One of the so-called non-Noether symmetries is defined by

$$\xi = q^2 \cos t, \qquad \tau = q \sin t, \tag{19}$$

η being determined by (7). It is straightforward to check that

$$F = C_1/C_2, \tag{20}$$

where C_1 and C_2 are the linear first integrals

$$C_1 = q \cos t - \dot{q} \sin t, \qquad C_2 = q \sin t + \dot{q} \cos t, \tag{21}$$

is a constant of the motion satisfying property (16) with respect to the symmetry (19). Hence we can ask whether a Lagrangian exists for which (19) becomes a Noether symmetry and (20) its corresponding invariant. If L is such a Lagrangian, we must have a relation like (17), which here becomes

$$\partial^2 L/\partial \dot{q}^2 = 1/C_1 C_2^2. \tag{22}$$

So, all we have to do is find a particular solution of (22) (considered as a partial differential equation for L), and then see whether adding appropriate linear terms in $\dot{q}$ can actually provide us with a Lagrangian for the given equation (18). Integrating equation (22) twice, one easily obtains the particular solution

$$L(t, q, \dot{q}) = q^{-2}(q \cos t - \dot{q} \sin t) \ln\left(\frac{q \cos t - \dot{q} \sin t}{q \sin t + \dot{q} \cos t}\right), \tag{23}$$

and it is straightforward to check that the Euler–Lagrange equation computed from (23) is indeed equivalent to equation (18).

4. General results

We show now that the construction made in the previous section will always work for systems with one degree of freedom (of course in some open domain where all functions are assumed to be sufficiently smooth). So consider an arbitrary second-order equation

$$\ddot{q} = \Lambda(t, q, \dot{q}), \tag{24}$$

with corresponding vector field Γ as in (2).

Theorem 1. Let Y be a general symmetry (in the representation (10)) of Γ, and F a constant of the motion (i.e. $\Gamma(F)=0$) satisfying condition (16). Then there exists a Lagrangian L for equation (24), such that Y becomes a Noether symmetry with respect to L, and F its corresponding Noether invariant. Such an L follows from integrating the equation

$$(\partial^2 L/\partial \dot{q}^2)\mu = -\partial F/\partial \dot{q}, \tag{25}$$

adding if necessary a suitable linear part in $\dot{q}$.

Proof. We have the identity

$$\Gamma \frac{\partial}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} \Gamma - \frac{\partial}{\partial q} - \frac{\partial \Lambda}{\partial \dot{q}} \frac{\partial}{\partial \dot{q}}. \tag{26}$$

Computing Γ of both sides of (25), making use of the identity (26), taking account of $\Gamma(F)=0$ and of the symmetry requirement $\nu=\Gamma(\mu)$, we obtain

$$\mu \frac{\partial}{\partial \dot{q}}\left[\Gamma\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] + \nu \frac{\partial^2 L}{\partial \dot{q}^2} = \frac{\partial F}{\partial q}. \tag{27}$$

Next, multiplying (25) by $-\nu$, (27) by μ, and adding the results, we obtain in view of $Y(F)=0$

$$\frac{\partial}{\partial \dot{q}}\left[\Gamma\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] = 0,$$

or $\Gamma(\partial L/\partial \dot{q}) - \partial L/\partial q = l(t, q)$ for some function $l(t, q)$. Putting

$$L' = L + a(t, q)\dot{q} + b(t, q), \tag{28}$$

and choosing a and b in such a way that

$$\partial a/\partial t - \partial b/\partial q + l = 0 \tag{29}$$

(different solutions only differ by gauge terms), we have

$$\Gamma(\partial L'/\partial \dot{q}) - \partial L'/\partial q \equiv 0,$$

which expresses that the Euler–Lagrange equation computed from L' will be equivalent to the given equation (24). With L', introduce the one-form θ' as in (3). We then have

$$\langle \partial/\partial \dot{q}, i_Y \, \mathrm{d}\theta' - \mathrm{d}F \rangle = 0,$$

which is just another way of writing equation (25) for L'. Moreover, we know that $\Gamma(F)=0$, and that $\nu=\Gamma(\mu)$. Hence the lemma of § 2 implies $i_Y \, \mathrm{d}\theta' = \mathrm{d}F$, which expresses that Y is a Noether symmetry with respect to L', corresponding to the constant of the motion F.

Remarks

(i) Y of course need not be a point symmetry in the above general result.

(ii) Different F's with the same Y will in general yield different Lagrangians.

(iii) In the above proof, we have only used the condition $\nu=\Gamma(\mu)$, and not the other condition $Y(\Lambda)=\Gamma(\nu)$, which is needed to ensure that Y is a symmetry of Γ. Since in our conclusion Y turns out to be a Noether symmetry (and hence certainly a symmetry of Γ), theorem 1 implicitly contains the statement: If $Y = \mu \, \partial/\partial q + \nu \, \partial/\partial \dot{q}$ is a vector field with property $\nu = \Gamma(\mu)$, and if there exists a constant of the motion F ($\Gamma(F) = 0$) with the property $Y(F) = 0$, then Y is a symmetry of Γ, i.e. Y automatically satisfies the requirement $Y(\Lambda) = \Gamma(\nu)$. This at first sight surprising statement becomes less surprising if we note that it can easily be proved in a direct way from the general relation

$$[Y, \Gamma] = (\nu - \Gamma(\mu))\, \partial/\partial q + (Y(\Lambda) - \Gamma(\nu))\, \partial/\partial \dot{q}.$$

(iv) Recall that if Y is a symmetry corresponding to the first integral F via the rule $Y(F) = 0$, then ϕY is another symmetry with the same property, if ϕ itself is another constant of Γ (Sarlet and Cantrijn 1981a). If we start our analysis with ϕY instead of Y, the basic equation becomes

$$(\partial^2 L/\partial \dot{q}^2)\phi\mu = -\partial F/\partial \dot{q}.$$

Hence, if L_0 is a Lagrangian computed from Y and F, and L_1 a Lagrangian computed from ϕY and F, it is clear that we will have

$$\partial^2 L_0/\partial \dot{q}^2 = \phi\ \partial^2 L_1/\partial \dot{q}^2. \tag{30}$$

It is a known result (Currie and Saletan 1966) that such a relationship is precisely the necessary and sufficient condition for having two equivalent Lagrangians in one dimension (apart from the trivial equivalence through gauge terms). So, the freedom of passing from Y to ϕY will here exhaust all possibilities for finding equivalent Lagrangians. If in the known expression $\partial F/\partial \dot{q}$ another constant of the motion is recognised as a factor, making use of this freedom could simplify the calculation of a Lagrangian. In general, however, knowing two independent constants for a system with one degree of freedom is a quite optimistic assumption. Nevertheless, it is instructive to illustrate this possibility on our harmonic oscillator example of § 3. There we had

$$\partial F/\partial \dot{q} = -q/C_2^2,$$

so that passing from Y to $(-C_2^{-2})Y$ will produce the equation $\partial^2 L/\partial \dot{q}^2 = 1$, which obviously will lead to the familiar Lagrangian $L = \frac{1}{2}(\dot{q}^2 - q^2)$.

Let us come back now to theorem 1 which, for the part related to existence of a Lagrangian, can be summarised schematically as follows:

$$\left.\begin{array}{l} [Y, \Gamma] = 0 \\ \Gamma(F) = 0 \\ Y(F) = 0 \end{array}\right\} \to L.$$

This naturally suggests the new question: with known symmetry Y of Γ, does there exist a constant F with the above property, and if so, how can we find it? The following theorem provides the answer to this question.

Theorem 2. Let $Y = \mu\ \partial/\partial q + \nu\ \partial/\partial \dot{q}$ be a general symmetry of Γ; then there exists a constant of the motion F of Γ, satisfying $Y(F) = 0$. It is of the form

$$F = F(t, \sigma(t, q, \dot{q})), \tag{31}$$

where $\sigma(t, q, \dot{q}) = \text{constant}$ is a complete solution of the first-order equation

$$\frac{\mathrm{d}\dot{q}}{\mathrm{d}q} = \frac{\nu}{\mu}(t, q, \dot{q}), \qquad t \text{ treated as parameter}, \tag{32}$$

and $F(t, \sigma) = \text{constant}$ is a complete solution of the first-order equation

$$d\sigma/dt = f(\sigma, t), \qquad f(\sigma, t) \equiv \Gamma(\sigma). \tag{33}$$

Clearly, such an F is unique in the sense that all other solutions are functions of the one described above.

Proof. We start by considering the requirement $Y(F) = 0$. Its characteristic equations are

$$dq/\mu = d\dot{q}/\nu = dt/0.$$

Hence, its general solution is as described by (31) and (32). Obviously, it is assumed that $Y \not\equiv 0$, which implies $\mu \not\equiv 0$, so that equation (32) makes sense in some open domain. Next, in order that an F of the form (31) be a constant of the motion of Γ, we must have

$$\partial F/\partial t + (\partial F/\partial \sigma)\Gamma(\sigma) = 0. \tag{34}$$

This requirement can be met if and only if $\Gamma(\sigma)$ can be expressed as a function of t and σ alone, hence if and only if the following Jacobian determinant vanishes identically:

$$\det[\partial(\Gamma(\sigma), \sigma)/\partial(q, \dot{q})] \equiv 0. \tag{35}$$

Computing condition (35) explicitly, dividing by $(\partial\sigma/\partial\dot{q})^2$ (which is not identically zero), and putting

$$v = \frac{\partial\sigma/\partial q}{\partial\sigma/\partial\dot{q}},$$

an appropriate recombination of terms yields the equation

$$\frac{\partial v}{\partial t} + \dot{q}\frac{\partial v}{\partial q} + \Lambda\frac{\partial v}{\partial \dot{q}} + \frac{\partial\Lambda}{\partial q} - \frac{\partial\Lambda}{\partial\dot{q}}v - v^2 = 0. \tag{36}$$

Now $Y(\sigma) = 0$ implies that we have $v = -\nu/\mu$, and consequently

$$\Gamma(v) = (1/\mu^2)(\nu\Gamma(\mu) - \mu\Gamma(\nu)).$$

Finally, Y being a symmetry of Γ, we have the relations (12) and (13), so that the above relation becomes

$$\Gamma(v) \equiv \frac{\nu^2}{\mu^2} - \frac{Y(\Lambda)}{\mu} \equiv v^2 - \frac{\partial\Lambda}{\partial q} + \frac{\partial\Lambda}{\partial\dot{q}}v,$$

which shows that condition (36) is indeed identically satisfied. $\Gamma(\sigma)$ being some function of the form $f(\sigma, t)$, the complete statement of the theorem now follows from equation (34) via the method of characteristics.

Corollary. Given a general symmetry Y of Γ, there exists a Lagrangian L for equation (24), such that Y is a Noether symmetry with respect to L. This statement trivially follows from theorems 1 and 2.

Remarks

(a) The construction of a first integral F according to theorem 2 generalises similar results which were discussed by Prince (1980) for point symmetries, in the context of Lie's method of extended groups. Similarly, equation (33) represents the reduction of the given second-order equation to a first-order one through a symmetry Y, in the same way as is usually done for point symmetries only (see e.g. Bluman and Cole (1974)).

(b) The combined results of theorems 1 and 2 constitute a generalisation of ideas presented previously (Sarlet 1978), but in a discussion which was limited to some elementary symmetries, motivated by earlier work of Denman (1965, 1966).

(c) The above proof not only shows the existence of F, but also the way to compute it in the given coordinates. The mere existence of F can most easily be established as follows. Since Y and Γ are commuting vector fields in three dimensions, there exist new local coordinates (x, y, z) such that $\Gamma = \partial/\partial x$ and $Y = \partial/\partial y$. Hence, arbitrary functions of z will satisfy all requirements. We are indebted to the referee for pointing out this useful clarification.

For the sake of completeness, let us also present at least one way in which a suitable symmetry Y could be computed, if only F is known. The following result can easily be checked by direct verification.

Theorem 3. Let F be a constant of the motion of Γ, and g a particular solution of the equation

$$\Gamma(g) = \partial\Lambda/\partial\dot{q}. \tag{37}$$

Then a symmetry of Γ with property $Y(F) = 0$ is determined by the components

$$\mu = (\partial F/\partial\dot{q}) \exp(g), \qquad \nu = -(\partial F/\partial q) \exp(g). \tag{38}$$

5. Other examples and applications

In § 3, we have turned one of the 'non-Noether' point symmetries for the harmonic oscillator into a Noether symmetry with respect to an unusual Lagrangian. It is a simple exercise to derive similar results for the other two 'non-Noether' symmetries, which can be associated with the same first integral $F = C_1/C_2$.

5.1. Damped harmonic oscillator

Consider the equation

$$\ddot{q} = -2\gamma\dot{q} - \omega_0^2 q. \tag{39}$$

The time-translation symmetry,

$$\xi = 0, \quad \tau = 1 \rightarrow \mu = -\dot{q},$$

can be associated with the first integral

$$F = \tfrac{1}{2}\ln[\omega^2 q^2 + (\dot{q} + \gamma q)^2] - (\gamma/\omega)\tan^{-1}[(\dot{q} + \gamma q)/\omega q],$$

where $\omega^2 = \omega_0^2 - \gamma^2$ is assumed to be positive. Twice integrating equation (25) for this case easily yields the particular solution

$$L = \rho\tan^{-1}\rho - \tfrac{1}{2}\ln(1 + \rho^2), \qquad \rho \equiv (\omega q)^{-1}(\dot{q} + \gamma q).$$

This is not yet a good Lagrangian for the problem. But following the procedure outlined in theorem 1, one easily finds that a Lagrangian is given by

$$L' = L - \ln(\omega q).$$

Equation (39) also has the symmetry $\xi = q$, $\tau = 0$, which can be associated with the first integral

$$F = \tan^{-1} \rho + \omega t,$$

and eventually leads to the Lagrangian

$$L = -\omega\rho \tan^{-1} \rho + \tfrac{1}{2}\omega \ln(1+\rho^2) - \omega^2 t\rho.$$

These results confirm those obtained by Sarlet (1978).

We would like now to illustrate on this example another way in which the ideas of the previous section could find applications. As said before, when only an F is known, it can in general be difficult to find (e.g. via theorem 3) a corresponding Y. Yet it can happen that a class of symmetry transformations for a given differential equation can be identified very easily, and then it becomes a simple matter to check whether, among the infinitesimal generators of these transformations, a Y exists corresponding to the given F.

For the damped harmonic oscillator, one easily recognises the following class of one-parameter families of symmetry transformations $(t, q) \leftrightarrow (T, Q)$,

$$t = T + h(\lambda), \qquad q = Q \exp[g(\lambda)], \tag{40}$$

where $h(0) = 0$, $g(0) = 0$, but h and g are further arbitrary. The corresponding infinitesimal generators are obtained by computing $\mathrm{d}/\mathrm{d}\lambda$ at $\lambda = 0$. Setting $c = h'(0)$, $a = g'(0)$, we obtain

$$\xi = aq,\ \tau = c \rightarrow \mu = aq - c\dot{q}. \tag{41}$$

Consider now the constant of the motion

$$F = \tfrac{1}{2} \exp(2\gamma t)(\dot{q}^2 + 2\gamma q\dot{q} + \omega_0^2 q^2)$$

(see e.g. Bahar and Kwatny (1981)).

Requiring $Y(F) = 0$ for a Y of type (41) immediately yields the condition $a + c\gamma = 0$. Choosing $c = 1$, $a = -\gamma$, equation (25) becomes

$$\partial^2 L/\partial \dot{q}^2 = \exp(2\gamma t),$$

which leads to the familiar Lagrangian

$$L = \tfrac{1}{2} \exp(2\gamma t)(\dot{q}^2 - \omega_0^2 q^2). \tag{42}$$

5.2. General autonomous equations

Consider for example the equation

$$\ddot{q} = -k\dot{q}^2,$$

which has the first integral $F = \dot{q} \exp(kq)$. It can obviously be associated with the time-translation symmetry of the differential equation, and in this way gives rise to the Lagrangian

$$L = \dot{q}(1 - \ln \dot{q}) \exp(kq).$$

Now a general autonomous equation,

$$\ddot{q} = \Lambda(q, \dot{q}), \tag{43}$$

of course always has the time-translation symmetry

$$Y = -\dot{q}\, \partial/\partial q - \Lambda(q, \dot{q})\, \partial/\partial \dot{q} \tag{44}$$

(equivalent to $Y = \partial/\partial t$ via addition of Γ).

Theorem 2 states that such an equation certainly has a time-independent first integral $F(q, \dot{q})$. It is interesting to refer here to a paper by Kobussen (1979), in which the author discusses the inverse problem for autonomous equations like (43). His main result is that the Lagrangian for such an equation is of the form

$$L = \dot{q} \int^{\dot{q}} y^{-2} F(q, y)\, \mathrm{d}y. \tag{45}$$

Computing the second derivative of (45) with respect to $\dot{q}$, we find

$$\partial^2 L/\partial \dot{q}^2 = (1/\dot{q})\, \partial F/\partial \dot{q},$$

which is precisely our equation (25) related to the symmetry (44). Theorem 1 therefore generalises Kobussen's treatment, in the sense that his results can be understood as belonging to the particular case of time-translation symmetry.

6. Concluding remarks

We have tried to complete the 'open base' of the triangular diagram of figure 1, which is suggested by Noether's theorem in classical mechanics. For the time being, our analysis has been restricted to the simple case of systems with one degree of freedom. Some of the arguments we have used certainly will carry over to the multiple degree of freedom case. Let us refer for example to Lutzky (1979), who has briefly discussed the problem of finding an appropriate first integral for given symmetry, not necessarily of point type. Still it remains true that the case of one degree of freedom is substantially different from the case $n > 1$. Theorem 1 can certainly not be generalised in the same terms to cases where $n > 1$. To illustrate this, it suffices to make the following observation about equation (17) with L unknown. For $n = 1$, once a particular solution is known, other solutions can differ from this one by at most linear terms in $\dot{q}$, and this played a role in our theorem 1. For general n, however, the homogeneous equation will have solutions not satisfying $\partial^2 L/\partial \dot{q}^i\, \partial \dot{q}^j = 0$. Nevertheless, we hope to present some analogous considerations for general n in a forthcoming paper. There seem to be two possible directions in which to go: either, the knowledge of one Y and corresponding F could at least reduce the problem of construction of a Lagrangian, or the existence of such a Lagrangian could be ensured if a sufficiently large Lie algebra of symmetries with corresponding F's is known. In this respect, we have to mention the important work by Takens (1977), who has constructed a very general abstract mathematical theory, in which questions similar to the one treated here have been solved. We have been using much more elementary techniques, and it would be a non-trivial problem to 'translate' Takens' results to the particular situation discussed in this paper, because already the notions of symmetry and conserved quantity are quite differently defined here.

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